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Approximations to sloshing frequencies for rectangular tanks with internal structures

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Abstract. The frequencies of free oscillation of a fluid in a rectangular tank are reduced by the introduction of a rigid internal structure. This paper gives general, approximate methods for the calculation of the oscillation frequencies when the structure is a cylinder of arbitrary cross section spanning the tank, and with generators normal to one pair of vertical faces. Particular results are given for submerged, circular cylinders and both vertical and horizontal thin baffles.

1. Introduction

A consequence of the theorems on eigenvalue problems described by Courant and Hilbert [1, Chapter VI] is that a rigid structure introduced into a tank of fluid will reduce the frequencies of free oscillations as long as the free surface is unchanged. A number of recent papers have investigated such changes for two-dimensional motions in a rectangular container of fluid. Evans and McIver [2] considered the case of a thin vertical baffle introduced into the tank, from either above or below, while methods for structures of finite thickness are given by Watson and Evans [3] and they present calculations for rectangular blocks and submerged circular cylinders. Davis [4] reconsidered some of the problems investigated by Watson and Evans and gives alternative methods of solution.

In all of the work mentioned above the fluid motion is assumed to be two dimensional; the current work gives approximate methods for equivalent three dimensional motions. The geometries considered are where a cylinder of arbitrary cross section spans a rectangular tank so that the cylinder generators are normal to two opposite vertical faces of the tank. In particular, results are given for circular cylinders and thin baffles. The methods given may also be used for some of the two-dimensional problems discussed in references [2–3].

In Section 2 a simple approach is described for obtaining approximate solutions when a typical dimension of the cylinder cross section is small compared to the length scale of the fluid motion. A similar method has been used in other cavity resonance problems; for example, Davidovitz and Lo [5] calculate cut-off wavenumbers in electromagnetic wave guides. Here, the procedure is illustrated for two-dimensional motions when a submerged circular cylinder is introduced into a rectangular tank and for three-dimensional motions in the present of a thin baffle. For three-dimensional motions involving structures of non-zero cross-sectional area much more care is required and an alternative procedure is given in Section 3. This is based on a scheme of matched asymptotic expansions and is similar to that used by McIver [6] for fluid oscillations in channels. An expression is derived for the natural oscillation frequencies

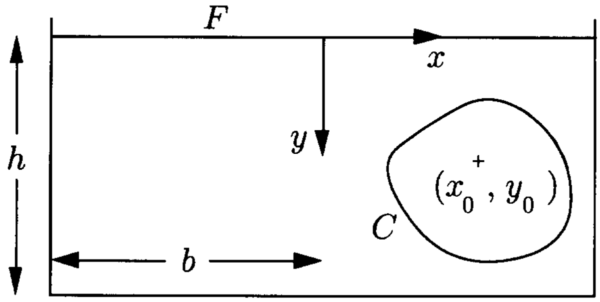

Fig. 1. Definition sketch.

of fluid in a rectangular tank containing a submerged cylinder of arbitrary cross section. The approximations are compared with numerical solutions in Section 4.

2. Formulation and small structure approximation

This paper is concerned with sloshing in a rectangular tank of uniform depth h. Cartesian coordinates are chosen with $y = 0$ in the mean free surface and the fluid occupies $|x| < b$, $0 < y < h$, $|z| < d$. Under the usual assumptions of the linearised theory of water waves, the flow may be described by a velocity potential satisfying Laplace's equation in the fluid region. Bodies occupying $|z| < d$ and with generators parallel to the z direction will be introduced into the tank and a cross section is illustrated in Fig. 1. In this cross section, a reference point within the body has coordinates $(x, y) = (x_0, y_0)$, the body surface is denoted by C and the free surface by F. The boundary conditions of no flow through the walls at $z = \pm d$ may be satisfied by factoring out from the potential a z-dependence $\cos p(z - d)$, where $p = L\pi/2d$ and L is any integer. Time-harmonic motions of the fluid may therefore be described by a potential of the form

$$\Phi(x, y, z, t) = \phi(x, y) \cos p(z - d) \cos \omega t \tag{2.1}$$

where

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} - p^2 \phi = 0 \tag{2.2}$$

in the fluid. The potential ϕ must also satisfy the linearised free-surface condition

$$K\phi + \frac{\partial \phi}{\partial y} = 0 \text{ on } y = 0, \ |x| < b, \tag{2.3}$$

where $K = \omega^2/g$, and the condition of no flow through the solid boundaries

$$\frac{\partial \phi}{\partial n} = 0 \text{ on } C, \quad \{|x| = b, 0 < y < h\} \text{ and } \{y = h, |x| < b\}. \tag{2.4}$$

where n is a normal coordinate measured into the fluid.

It is easily verified that solutions of the above problem when there is no body within the tank are of the form

$$\phi_M = \cos \alpha_M (x - b) \cosh k_M (y - h), \tag{2.5}$$

where

$$\alpha_M = M\pi/2b, \quad k_M = (\alpha_M^2 + p^2)^{1/2} \tag{2.6}$$

and M is any integer. From the free-surface condition (2.3) the corresponding value of the frequency parameter K is

$$K_M = k_M \tanh k_M h. \tag{2.7}$$

If M is odd these modes of oscillation are antisymmetric in x while if M is even the modes are symmetric in x. For a given tank geometry there are a doubly infinite set of modes which can be identified by the integers L and M in the definitions of p and α_M.

The aim is to calculate the change in K when a body of uniform cross-section spans the tank in the z direction. First of all a simple approach is suggested. Applying Green's theorem over the boundaries of the fluid domain to the potentials with and without the body, ϕ and ϕ_M respectively, gives

$$\int_F \left(\phi \frac{\partial \phi_M}{\partial n} - \phi_M \frac{\partial \phi}{\partial n}\right) \mathrm{d}s + \int_C \phi \frac{\partial \phi_M}{\partial n}\, \mathrm{d}s = 0, \tag{2.8}$$

where the body boundary conditions (2.4) have been used to eliminate a number of terms. Replacing the normal derivatives on F using the free-surface condition (2.3) and rearranging yields

$$K = K_M - \int_C \phi \frac{\partial \phi_M}{\partial n}\, \mathrm{d}s \Big/ \int_F \phi\phi_M \, \mathrm{d}s. \tag{2.9}$$

When a typical dimension of C is much smaller than the cross-sectional length scale of the fluid motion, equation (2.9) may be used, with care, to estimate K by observing that over most of the fluid domain the solution will differ little from that given in equations (2.5–7). As an illustration, suppose C is a circle of radius a centred on (x_0, y_0) and the motion is two-dimensional, that is $p = 0$. A typical length scale of the fluid motion is k_M^{-1} so that an approximation is sought under the assumption $k_M a \ll 1$. Define plane polar coordinates (r, θ) by

$$x - x_0 = r \sin\theta, \quad y - y_0 = r\cos\theta. \tag{2.10}$$

Near (x_0, y_0) the unperturbed potential may be expanded as

$$\begin{aligned}\phi_M &= \cos k_M(x_0 - b)\cosh k_M(y_0 - h) - k_M r \sin\theta \sin k_M(x_0 - b)\cosh k_M(y_0 - h) \\ &\quad + k_M r\cos\theta\cos k_M(x_0 - b)\sinh k_M(y_0 - h) + O((k_M r)^2).\end{aligned} \tag{2.11}$$

The terms in (2.11) are solutions of Laplace's equation and represent a local uniform flow. Providing $k_M a$ is small the perturbation to the motion from introducing C will be negligible except in the immediate vicinity of the body. Near the body, ϕ is written

$$\begin{aligned}\phi &= \text{constant} - k_M\left(r + \frac{a^2}{r}\right)\sin\theta\sin k_M(x_0 - b)\cosh k_M(y_0 - h) \\ &\quad + k_M\left(r + \frac{a^2}{r}\right)\cos\theta\cos k_M(x_0 - b)\sinh k_M(y_0 - h) + O((k_M r)^2)\end{aligned} \tag{2.12}$$

which has zero normal derivative on C and for large r/a the uniform flow terms correspond with those in the unperturbed potential (2.11). The value of the constant in (2.12) is not needed for the present calculation as it does not contribute to the integrals in (2.9). The integral over C is approximated using the forms of ϕ and ϕ_M given by the terms displayed explicitly in equations (2.11) and (2.12). The influence of the body is small in the free-surface integral and ϕ is approximated by ϕ_M over F using (2.4). With these approximations to the potentials, (2.9) yields

$$K \sim K_M - \frac{2\pi k_M^2 a^2}{b\cosh^2 k_M h}\left(\sin^2 k_M(x_0-b)\cosh^2 k_M(y_0-h) + \cos^2 k_M(x_0-b)\sinh^2 k_M(y_0-h)\right) \tag{2.13}$$

with $k_M = M\pi/2b$.

The above calculation can be adapted to produce the correct leading-order approximation to K for non-zero p, but this is not straightforward. Any structure with non-zero area of cross section forces source-like terms in the local flow when $p \neq 0$. As will be seen in the following section, finding the leading-order changes in the frequency then involves taking further terms in the expansion (2.11) and choosing a suitable value for the constant in (2.12). To treat this case with more confidence, to formalise the above and to generalise to arbitrary contours C another approach is presented in Section 3.

For thin baffles having negligible cross-sectional area, so that the source terms mentioned in the previous paragraph are not present, there is no difficulty in applying the above procedure for non-zero p. For example, consider a surface-piercing baffle occupying $0 < y < a$ at $x = x_0$. Near $(x, y) = (x_0, 0)$, the unperturbed potential ϕ_M in equation (2.5) may be expanded as

$$\begin{aligned}\phi_M &= \cos\alpha_M(x_0-b)\cosh k_M h - \alpha_M(x-x_0)\sin\alpha_M(x_0-b)\cosh k_M h \\ &\quad - k_M y\cos\alpha_M(x_0-b)\sinh k_M h + O((\alpha_M(x-x_0))^2, (k_M y)^2)\end{aligned} \tag{2.14}$$

and therefore locally there is a combination of horizontal and vertical uniform flows. The free-surface condition (2.3) is approximated locally by a rigid-lid condition. The constant and vertical flow terms in (2.14) do not contribute to the integral over C in (2.9) because of their symmetry about $x = x_0$. For the horizontal flow term, the corresponding local potential that has zero normal derivative on the baffle and its image in $y = 0$, and recovers the term proportional to $(x - x_0)$ in (2.14) at large distances, is

$$\phi = \Re\left\{-\alpha_M \sin\alpha_M(x_0-b)\cosh k_M h[(x-x_0+iy)^2 + a^2]^{1/2}\right\}. \tag{2.15}$$

This form of ϕ is used in the integral over C in (2.9) while over the free surface ϕ is approximated by ϕ_M, under the assumption that the disturbance created by the baffle is small. This results in

$$K = K_M - \frac{\pi a^2\alpha_M^2}{2b}\sin^2\alpha_M(x_0-b). \tag{2.16}$$

This result has been confirmed and extended to higher order using the formal solution method of Section 3 by Jeyakumaran [7].

For a bottom-mounted baffle occupying $a < y < h$ at $x = x_0$, the result corresponding to (2.16) is

$$K = K_M - \frac{\pi a^2\alpha_M^2}{2b\cosh^2 k_M h}\sin^2\alpha_M(x_0-b) \tag{2.17}$$

and for a horizontal baffle occupying $b - a < x < b$ at $y = y_0$

$$K = K_M - \frac{\pi a^2 k_M^2}{2b\cosh^2 k_M h} \sinh^2 k_M(y_0 - h). \tag{2.18}$$

3. The submerged cylinder by matched asymptotic expansions

The contour C is taken to be fully submerged with the typical dimension a much less than the minimum distance of C from the boundaries (including the free surface). The boundary value problem to be solved is given by equations (2.2–4). Define

$$\alpha = (k^2 - p^2)^{1/2} \tag{3.1}$$

where k is the positive real root of

$$K = k \tanh kh. \tag{3.2}$$

As may be seen from the solution to the unperturbed problem, equations (2.5–7), or the eigenfunction representations of the multipoles given in Appendix A, α and k are the natural parameters to describe the horizontal and vertical variations in the wave motion. In the absence of C, $\alpha = \alpha_M \equiv M\pi/2b$ where M is an integer. For the perturbed problem with C introduced into the tank, define

$$\sigma = M\pi - 2\alpha b. \tag{3.3}$$

The aim is to determine an approximation to σ, and hence the change in K resulting from the introduction of C, under the assumption that a is small compared with all other length scales. Define $\varepsilon = a/h$, assumed to be small, and put

$$\sigma = f(\varepsilon)\sigma_2 + \ldots, \tag{3.4}$$

where $f(\varepsilon) \ll 1$ and $\sigma_2 = O(1)$. The form of $f(\varepsilon)$ is to be determined (it turns out that $f(\varepsilon) = \varepsilon^2$, hence the adoption of the subscript 2 in (3.4)). Substituting (3.3) and (3.4) into (3.1) gives

$$k = k_M \left(1 - f(\varepsilon)\frac{M\pi\sigma_2}{4k_M^2 b^2} + \cdots\right) \tag{3.5}$$

and so from (3.2)

$$K = K_M(1 - f(\varepsilon)V + \cdots), \tag{3.6}$$

where

$$V = \frac{M\pi\sigma_2}{4k_M^2 b^2}\left(1 + \frac{k_M^2 h}{K_M} - K_M h\right). \tag{3.7}$$

Initial development of inner and outer solutions

The solution is by the method of matched asymptotic expansions. Inner and outer regions will be defined and corresponding solutions developed which are only fully determined when matching in an overlap region has been carried out. The initial aim is to obtain the leading-order non-constant terms in the inner and outer solutions.

In the outer region, at distances $r \gg a$ from C, a non-dimensional radial coordinate is defined by

$$R = r/h, \tag{3.8}$$

where the polar coordinates (r, θ) are defined by (2.10). The complete outer solution $\Psi(R, \theta) \equiv \phi(r, \theta)$ is expressed as

$$\Psi = \hat{A}_0 g_0(R, \theta) + \sum_{n=1}^{\infty} (\hat{A}_n g_n(R, \theta) + \hat{B}_n h_n(R, \theta)) \tag{3.9}$$

where

$$g_n = \sin \sigma\, \phi_n^{(b)} \quad \text{and} \quad h_n = \sin \sigma\, \psi_n^{(b)} \tag{3.10}$$

and $\phi_n^{(b)}$ and $\psi_n^{(b)}$ are the multipole potentials defined in appendix A, that is they are singular solutions of the modified Helmholtz equation satisfying all the conditions of the problem except that on C. The additional factor of $\sin \sigma$ has been introduced for convenience. Write

$$g_n = g_{n,1} + \sin \sigma\, g_{n,2} \quad \text{and} \quad h_n = h_{n,1} + \sin \sigma\, h_{n,2} \tag{3.11}$$

where

$$g_{n,2} = K_n(\delta R) \cos n\theta \quad \text{and} \quad h_{n,2} = K_n(\delta R) \sin n\theta \tag{3.12}$$

are the singular parts of the multipoles and $\delta = ph$. Thus, for example, g_0 is a source while g_1 and h_1 are a horizontal and vertical dipole respectively. From the results in appendix A, part (b), the non-singular parts have expansions of the form

$$g_{n,1} = \sum_{q=0}^{\infty} (c_{nq} \cos q\theta + d_{nq} \sin q\theta) I_q(\delta R) \tag{3.13}$$

$$h_{n,1} = \sum_{q=0}^{\infty} (e_{nq} \cos q\theta + f_{nq} \sin q\theta) I_q(\delta R). \tag{3.14}$$

In the above K_n and I_q denote modified Bessel functions. By virtue of (3.3) and (3.4), the expansion coefficients in equation (3.13) have expansions in terms of ε in the form

$$c_{nq} = c_{nq,0} + f(\varepsilon) c_{nq,2} + \cdots, \tag{3.15}$$

where $c_{nq,j} = O(1)$, with similar expansions for the remaining coefficients in (3.13–14). Note that the $O(1)$ terms in these coefficient expansions arise from the first terms of the summations over m in equations (A31–32).

From previous work on scattering by submerged bodies, for example Davis and Leppington [8], and from the related work of McIver [6] it is clear that the outer solution will contain only sources and dipoles at leading order. Thus, the leading-order outer solution is written

$$\Psi^{(0)} = A_0 g_{0,1}^{(0)} + A_1 g_{1,1}^{(0)} + B_1 h_{1,1}^{(0)} \tag{3.16}$$

where, in a standard notation, a superscript in parentheses is used to denote the order in ε of a quantity. (This form for $\Psi^{(0)}$ may be justified by retaining a full multipole expansion and allowing the matching to eliminate all but the source and dipole terms.) The problem is homogeneous so the order in ε of the solution may be freely chosen, it is natural to take it to be $O(1)$ as in (3.16).

For an inner region within distances $r \ll h$ of C, a radial inner coordinate is defined by

$$\rho = r/a. \tag{3.17}$$

In terms of the inner coordinates the inner solution $\psi(\rho,\theta) \equiv \phi(r,\theta)$ must satisfy the field equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\theta^2} - \varepsilon^2\delta^2\psi = 0 \tag{3.18}$$

and the boundary condition

$$\partial\psi/\partial n = 0 \ \text{ on } \ C. \tag{3.19}$$

The solution is fully determined by matching with the outer solution.

Using well-known expansions of the modified Bessel functions, the inner expansion of the leading-order outer solution, equation (3.16), is

$$\Psi^{(0,1)} = \Pi_0 + \Pi_1\varepsilon\rho\cos\theta + \Pi_2\varepsilon\rho\sin\theta, \tag{3.20}$$

where

$$\Pi_0 = A_0c_{00,0} + A_1c_{10,0} + B_1e_{10,0}, \tag{3.21}$$

$$\Pi_1 = \tfrac{1}{2}\delta(A_0c_{01,0} + A_1c_{11,0} + B_1e_{11,0}), \tag{3.22}$$

$$\Pi_2 = \tfrac{1}{2}\delta(A_0d_{01,0} + A_1d_{11,0} + B_1f_{11,0}). \tag{3.23}$$

Here, $\Psi^{(p,q)}$ denotes the result of expressing $\Psi^{(p)}$ in inner variables and expanding up to $O(\varepsilon^q)$. A similar notation is used for the inner solution. Thus, $\psi^{(q)}$ is the inner solution up to $O(\varepsilon^q)$ which when expressed in terms of outer variables and expanded to $O(\varepsilon^p)$ is denoted by $\psi^{(q,p)}$. The matching principle requires that $\psi^{(q,p)} \equiv \Psi^{(p,q)}$ when both are expressed in the same coordinates. See Crighton and Leppington [9] for further details of the matching principle.

Equation (3.20) suggests an inner development

$$\psi^{(1)} = P_0 + \varepsilon\{P_1 + P_2(\rho\cos\theta + \tau_1(\rho,\theta)) + P_3(\rho\sin\theta + T_1(\rho,\theta))\}, \tag{3.24}$$

where, from equations (3.18) and (3.19), τ_1 and T_1 are harmonic functions satisfying

$$\frac{\partial\tau_1}{\partial n} = -\frac{\partial}{\partial n}(\rho\cos\theta) \ \text{ and } \ \frac{\partial T_1}{\partial n} = -\frac{\partial}{\partial n}(\rho\sin\theta) \ \text{ on } \ C. \tag{3.25}$$

The potentials τ_1 and T_1 are respectively the disturbances to a uniform flow past C in the horizontal and vertical directions respectively. From Batchelor [10, p. 127], as $\rho \to \infty$

$$\tau_1 = \upsilon\frac{\cos\theta}{\rho} + \lambda\frac{\sin\theta}{\rho} + O(\rho^{-2}) \tag{3.26}$$

and

$$T_1 = \Upsilon\frac{\cos\theta}{\rho} + \Lambda\frac{\sin\theta}{\rho} + O(\rho^{-2}) \tag{3.27}$$

where the dipole coefficients v, λ, Υ and Λ are assumed known. The outer expansion of (3.24) when expressed in inner coordinates is therefore

$$\psi^{(1,0)} = P_0 + \varepsilon(P_2\rho\cos\theta + P_3\rho\sin\theta). \tag{3.28}$$

Matching (3.20) and (3.28) gives

$$P_0 = \Pi_0, \quad P_2 = \Pi_1, \quad P_3 = \Pi_2. \tag{3.29}$$

INNER AND OUTER SOLUTIONS TO $O(\varepsilon^2)$

The above matching still leaves undetermined the constants A_0, A_1 and B_1 which were introduced in equation (3.16). The inner and outer solutions are now extended to $O(\varepsilon^2)$ and the subsequent matching results in a set of homogeneous linear equations for these constants. The requirement that these equations have a non-trivial solution yields the desired frequencies of oscillation for the fluid.

Further expansion of the inner solution (3.24) using (3.26–27) gives

$$\begin{aligned}\psi^{(1,2)} &= \Pi_0 + \varepsilon\Big\{P_1 + \Pi_1\left(\frac{R\cos\theta}{\varepsilon} + \frac{\varepsilon}{R}(v\cos\theta + \lambda\sin\theta)\right)\\ &\quad + \Pi_2\left(\frac{R\sin\theta}{\varepsilon} + \frac{\varepsilon}{R}(\Upsilon\cos\theta + \Lambda\sin\theta)\right)\Big\}.\end{aligned} \tag{3.30}$$

The dipole terms in (3.30) appear at an $O(\varepsilon^2)$ higher than the uniform flow terms which can only be reconciled with the outer solution if the same is true in the inner expansions of the dipole potentials. From (3.11) this requires $\sigma = O(\varepsilon^2)$ and so the choice $f(\varepsilon) \equiv \varepsilon^2$ is made.

Retaining only the multipoles that can possibly match with (3.30) the outer solution can now be continued as

$$\begin{aligned}\Psi^{(2)} &= A_0 g_0^{(2)} + A_1 g_1^{(2)} + B_1 h_1^{(2)} + \varepsilon\{C_0 g_0^{(0)} + C_1 g_1^{(0)} + D_1 h_1^{(0)}\}\\ &\quad + \varepsilon^2\left\{E_0 g_0^{(0)} + \sum_{n=1}^{2}(E_n g_n^{(0)} + F_n h_n^{(0)})\right\}.\end{aligned} \tag{3.31}$$

Note that

$$\begin{aligned}g_n^{(2)} &= \sum_{q=0}^{\infty}[(c_{nq,0} + \varepsilon^2 c_{nq,2})\cos q\theta + (d_{nq,0} + \varepsilon^2 d_{nq,2})\sin q\theta]I_q(\delta R)\\ &\quad + \varepsilon^2\sigma_2 K_n(\delta R)\cos n\theta\end{aligned} \tag{3.32}$$

with a similar expression for $h_n^{(2)}$. The inner expansion of (3.31) yields

$$\begin{aligned}\Psi^{(2,2)} &= \Pi_0 + \varepsilon\left\{\Pi_3 + \Pi_1\rho\cos\theta + \Pi_2\rho\sin\theta + \frac{\sigma_2 A_1}{\delta}\frac{\cos\theta}{\rho} + \frac{\sigma_2 B_1}{\delta}\frac{\sin\theta}{\rho}\right\}\\ &\quad - \varepsilon^2\ln\varepsilon\,\sigma_2 A_0 + \varepsilon^2\Big\{\Pi_4 - \sigma_2 A_0\ln\rho + \Pi_5\rho\cos\theta + \Pi_6\rho\sin\theta + \Pi_7\rho^2\cos 2\theta\\ &\quad + \Pi_8\rho^2\sin 2\theta + \Pi_0\frac{1}{4}\delta^2\rho^2\Big\}.\end{aligned} \tag{3.33}$$

Only the form of this expansion is needed, the full details of the constants Π_m, $m = 3, 4, \ldots, 8$, are not required and so are omitted. The inner solution is continued as

$$\psi^{(2)} = \psi^{(1)} + \varepsilon^2 \ln \varepsilon \, P_4 + \varepsilon^2 \psi_2, \tag{3.34}$$

where $\psi^{(1)}$ is given by (3.24) and (3.29). The term at $O(\varepsilon^2 \ln \varepsilon)$ is chosen as a constant as this is the only harmonic function satisfying (3.19) that can match with (3.33). Substituting (3.34) into (3.18–19) and equating like terms in ε shows that ψ_2 must satisfy

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi_2}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi_2}{\partial \theta^2} = \delta^2 \Pi_0 \tag{3.35}$$

in the fluid and

$$\partial \psi_2 / \partial n = 0 \text{ on } C. \tag{3.36}$$

A particular solution of (3.35–36) is chosen in the form

$$\psi_{2,p} = \frac{1}{4}\delta^2 \Pi_0 \rho^2 + \Omega(\rho, \theta), \tag{3.37}$$

where $\Omega(\rho, \theta)$ is a harmonic function satisfying

$$\Omega + \delta^2 \Pi_0 \frac{S}{2\pi a^2} \ln \rho \to 0 \text{ as } \rho \to \infty \tag{3.38}$$

and the boundary condition

$$\frac{\partial \Omega}{\partial n} = -\frac{1}{4}\delta^2 \Pi_0 \frac{\partial \rho^2}{\partial n} \text{ on } C. \tag{3.39}$$

Here S is the area of the cross section C, the logarithmic term in (3.38) is due to the flux across C indicated by (3.39). Bearing in mind the inner expansion (3.33), the full form for ψ_2 is taken as

$$\begin{aligned} \psi_2 = {} & \psi_{2,p} + Q_0 + Q_1(\rho \cos\theta + \tau_1(\rho, \theta)) + Q_2(\rho \sin\theta + T_1(\rho, \theta)) \\ & + Q_3(\rho^2 \cos 2\theta + \tau_2(\rho, \theta)) + Q_4(\rho^2 \sin 2\theta + T_2(\rho, \theta)) \end{aligned} \tag{3.40}$$

which leads to an outer expansion for $\psi^{(2)}$ of

$$\begin{aligned} \psi^{(2,2)} = {} & \Pi_0 + \varepsilon \Big\{ P_1 + \Pi_1 \left(\rho \cos\theta + \upsilon \frac{\cos\theta}{\rho} + \lambda \frac{\sin\theta}{\rho} \right) \\ & + \Pi_2 \left(\rho \sin\theta + \Upsilon \frac{\cos\theta}{\rho} + \Lambda \frac{\sin\theta}{\rho} \right) \Big\} + \varepsilon^2 \ln \varepsilon \, \Pi_4 \\ & + \varepsilon^2 \Big\{ \frac{1}{4}\delta^2 \Pi_0 \rho^2 - \delta^2 \Pi_0 \frac{S}{2\pi a^2} \ln \rho + Q_0 + Q_1 \rho \cos\theta \\ & + Q_2 \rho \sin\theta + Q_3 \rho^2 \cos 2\theta + Q_4 \rho^2 \sin 2\theta \Big\}. \end{aligned} \tag{3.41}$$

Matching (3.41) and (3.33) gives, in particular,

$$\sigma_2 A_0 = \delta^2 \Pi_0 \frac{S}{2\pi a^2}, \qquad \frac{\sigma_2 A_1}{\delta} = \Pi_1 \upsilon + \Pi_2 \Upsilon, \qquad \frac{\sigma_2 B_1}{\delta} = \Pi_1 \lambda + \Pi_2 \Lambda. \tag{3.42}$$

Substituting for Π_i, $i = 0, 1, 2$, from (3.21–23) and expressing in matrix form gives

$$\begin{pmatrix} c_{00,0}(S/\pi a^2) & c_{10,0}(S/\pi a^2) & e_{10,0}(S/\pi a^2) \\ \upsilon c_{01,0} + \Upsilon d_{01,0} & \upsilon c_{11,0} + \Upsilon d_{11,0} & \upsilon e_{11,0} + \Upsilon f_{11,0} \\ \lambda c_{01,0} + \Lambda d_{01,0} & \lambda c_{11,0} + \Lambda d_{11,0} & \lambda e_{11,0} + \Lambda f_{11,0} \end{pmatrix} \begin{pmatrix} A_0 \\ A_1 \\ B_1 \end{pmatrix} = \frac{2\sigma_2}{\delta^2} \begin{pmatrix} A_0 \\ A_1 \\ B_1 \end{pmatrix}, \tag{3.43}$$

which is a standard eigenvalue problem to determine σ_2. The eigenvalues are a repeated value $\sigma_2 = 0$ and

$$\sigma_2 = \frac{\delta^2}{2} \left\{ c_{00,0} \frac{S}{\pi a^2} + (\upsilon c_{01,0} + \Upsilon d_{01,0}) \frac{c_{11,0}}{c_{01,0}} + (\lambda c_{01,0} + \Lambda d_{01,0}) \frac{e_{11,0}}{c_{01,0}} \right\}. \tag{3.44}$$

The zero eigenvalue leads to the remaining non-zero parts of the multipole potentials in (3.16) combining in such a way as to give a zero potential; the required result is given by (3.44).

The expansion coefficients appearing in (3.44) follow by comparison of (3.13–14) with (A31–32). From (A17) and (3.3)

$$\sinh 2\alpha_0 b = -i \sin 2\alpha b = i(-1)^M \sin \sigma \tag{3.45}$$

so that, bearing in mind (3.10),

$$\begin{aligned} c_{00,0} &= -i\Gamma_{00}((-1)^M \cos 2\alpha_M x_0 + 1) \cosh k_M(h - y_0), \\ c_{01,0} &= \frac{2ik_M}{p} \Gamma_{00}((-1)^M \cos 2\alpha_M x_0 + 1) \sinh k_M(h - y_0), \\ c_{11,0} &= \frac{2ik_M}{p} \Gamma_{10}((-1)^M \cos 2\alpha_M x_0 + 1) \sinh k_M(h - y_0), \\ d_{01,0} &= \frac{2i\alpha_M}{p} \Gamma_{00}(-1)^M \sin 2\alpha_M x_0 \cosh k_M(h - y_0), \\ e_{11,0} &= \frac{2k_M}{p} \Delta_{10}(-1)^M \sin 2\alpha_M x_0 \sinh k_M(h - y_0) \end{aligned} \tag{3.46}$$

and so

$$\begin{aligned} \sigma_2 = {} & \frac{\pi h}{2\alpha_M N_{0,M}^2} \left\{ \frac{Sp^2}{\pi a^2} (1 + (-1)^M \cos 2\alpha_M x_0) \cosh^2 k_M(h - y_0) \right. \\ & + \upsilon 2k_M^2 (1 + (-1)^M \cos 2\alpha_M x_0) \sinh^2 k_M(h - y_0) \\ & - (\Upsilon - \lambda) \alpha_M k_M (-1)^M \sin 2\alpha_M x_0 \sinh 2k_M(h - y_0) \\ & \left. + \Lambda 2\alpha_M^2 (1 - (-1)^M \cos 2\alpha_M x_0) \cosh^2 k_M(h - y_0) \right\}. \end{aligned} \tag{3.47}$$

4. Results

For the purposes of illustration, in this section attention will be focussed on two geometries of internal structure, the submerged circular cylinder and the surface-piercing vertical baffle. The first of these geometries was considered by Watson and Evans [3]. They employed a 'wide-spacing' approximation in which the cylinder is assumed to be far from the tank ends and the scattering properties of a submerged cylinder in open water are used. For a cylinder of radius a, the result for σ_2 in equation (3.47) applies with the cross-sectional area $S = \pi a^2$ and

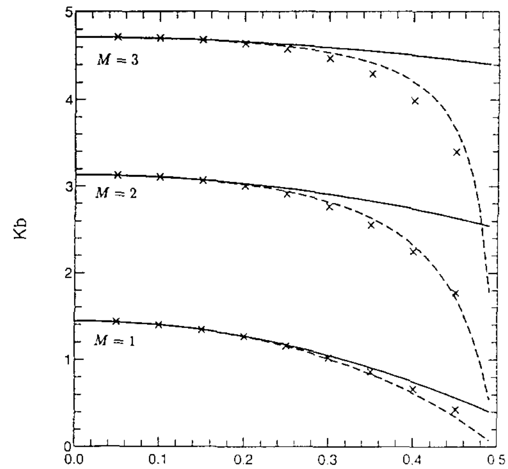

Fig. 2. Frequency parameter Kb v. radius a/b for a circular cylinder with centre at $(x_0/b, y_0/b) = (0, 0.5)$ in a tank of depth $h/b = 1$, long-tank wavenumber $pb = 0$. (———) equation (4.1), (– – – –), equation (4.2), (×××) exact solution.

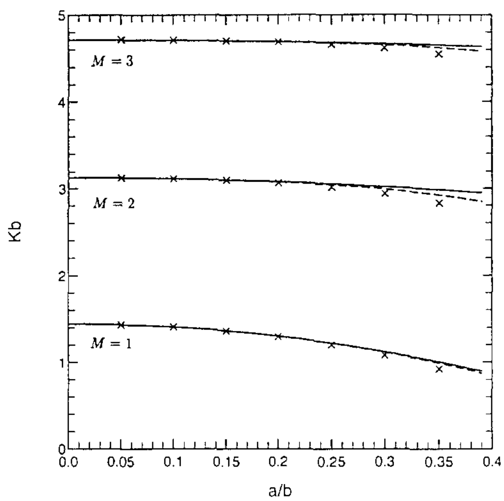

Fig. 3. Frequency parameter Kb v. radius a/b for a circular cylinder with centre at $(x_0/b, y_0/b) = (0, 0.6)$ in a tank of depth $h/b = 1$, long-tank wavenumber $pb = 0$. (———) equation (4.1), (– – – –), equation (4.2), (×××) exact solution.

the dipole coefficients, defined in equations (3.26–7), given by $\upsilon = \Lambda = 1$ and $\lambda = \Upsilon = 0$; the oscillation frequencies then follow from equation (3.6), namely

$$K = K_M(1 - \varepsilon^2 V), \tag{4.1}$$

which reduces to (2.13) when $p = 0$. Comparison is made with numerical solutions using a standard boundary element method [11] in Figs. 2–4. In each case the frequency parameter Kb is plotted as a function of non-dimensional cylinder radius a/b, where b is the tank half-width. These calculations, and others not presented here, suggest that equation (4.1) always works well provided the distance of the cylinder from the free surface is not less than the cylinder radius. In Fig. 3, the cylinder is sufficiently deeply submerged so that as the radius is increased the cylinder first touches the bottom rather than the free surface. In this, and similar cases,

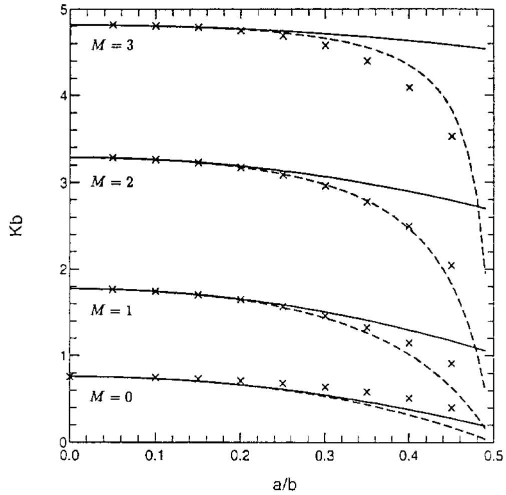

Fig. 4. Frequency parameter Kb v. radius a/b for a circular cylinder with centre at $(x_0/b, y_0/b) = (0.5, 0.5)$ in a tank of depth $h/b = 1$, long-tank wavenumber $pb = 1$. (———) equation (4.1), (– – – –), equation (4.2), (×××) exact solution.

equation (4.1) works well over almost the whole range of admissible a/b, although it should be noted that corresponding changes in Kb are not as significant.

A substantial improvement in the range of validity of the approximate theory can be obtained by using an equivalent rational form to (4.1). Write

$$\frac{K}{K_M} = \frac{1+\varepsilon^2 A}{1+\varepsilon^2 B} = 1 + \varepsilon^2(A-B) + O(\varepsilon^4) \tag{4.2}$$

which is equivalent to (4.1) provided

$$A - B = -V. \tag{4.3}$$

If it is now required that

$$K/K_M = C \tag{4.4}$$

at a prescribed value $\varepsilon = \varepsilon_c$, where C is to be chosen, then

$$B = -V/(C-1) - 1/\varepsilon_c^2 \tag{4.5}$$

and A follows from (4.3). It remains to choose ε_c and C. One of the geometries considered by Watson and Evans [3] was a bottom-mounted rectangular block. For two-dimensional motions, as the block is extended to the free surface the frequencies of all modes approach zero. Guided by this, ε_c is chosen to be the value of ε corresponding to the cylinder touching the surface and C is chosen to be zero for all modes. Results corresponding to the rational form are given by the dashed lines in Figs. 2–4 and a dramatic improvement in the accuracy of the theory is obtained. In the geometry of Fig. 3, the cylinder strictly cannot touch the free surface. However, the same rational approximation was used by allowing the cylinder to 'pierce' the bottom of the tank.

The results of Figs. 2–3 are for purely two-dimensional motions but Fig. 4 corresponds to three-dimensional motion, that is a non-zero value of the long-tank wave number p. It may be observed that for the lowest mode the best accuracy is obtained by using equation (4.1), rather

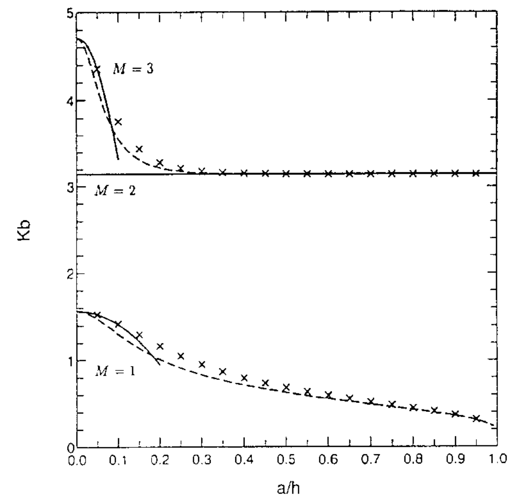

Fig. 5. Frequency parameter Kb v. length a/h for a surface piercing baffle at $x_0/b = 0$ in a tank of depth $h/b = 2$, long-tank wavenumber $pb = 0$. (———) equation (2.16), (- - - -), variational approximation, (×××) exact solution.

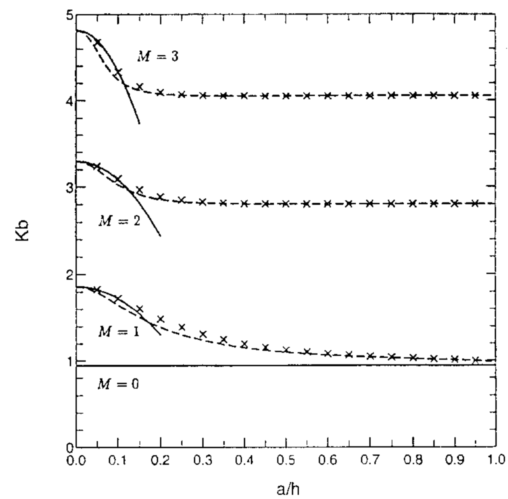

Fig. 6. Frequency parameter Kb v. length a/h for a surface piercing baffle at $x_0/b = 0.2$ in a tank of depth $h/b = 2$, long-tank wavenumber $pb = 1$. (———) equation (2.16), (- - - -), variational approximation, (×××) exact solution.

than its equivalent rational form, and this appears to be true for all cases with non-zero p. Note the lowest mode for non-zero p is symmetric for symmetric geometries. For two-dimensional motions when $p = 0$, this mode corresponds to $K = 0$ and the lowest mode with non-zero K is antisymmetric for symmetric geometries. The accuracy obtained in Figs. 2–4 by the approximate formulae is typical of the comparisons made by the authors.

Turning now to the surface-piercing vertical baffle, comparisons between the approximate formula (2.16) and accurate computations are made in Figs. 5–6. Here the frequency parameter is plotted against the baffle length to depth ratio a/h which always lies in the range zero to one. The accurate computations were made using the eigenfunction expansion method described for two-dimensional motions by Evans and McIver [2] and extended to non-zero p by Jeyakumaran [7]. In general, this approximation works well as long as the barrier submergence does not exceed a tenth of the depth. A rational form of this formula may be generated easily as the natural frequencies of the fully-divided tank, corresponding to $a/h = 1$, are known. However, the rational forms so obtained do not perform as well as (2.16) for small a/h and, furthermore, another good approximation for larger a/h is available from Evans and McIver [2]. Based on their eigenfunction method, Evans and McIver obtained a 'variational' approximation and this is easily extended to non-zero p. The result is that the frequencies of the free oscillations must satisfy

$$\frac{\sin 2\alpha b}{\sin\alpha(b+x_0)\sin\alpha(b-x_0)} = \frac{1}{c_{00}^2}\sum_{n=1}^{\infty} s_n c_{0n}^2 \tag{4.6}$$

where

$$s_n = \frac{\alpha}{\alpha_n}(\coth\alpha_n(b+x_0) + \coth\alpha_n(b-x_0)), \tag{4.7}$$

$$c_{0n} = \frac{1}{h}\int_L \psi_0(y)\psi_n(y)\,\mathrm{d}y, \tag{4.8}$$

$$\psi_n(y) = \cos k_n(h-y)/[\tfrac{1}{2}(1+\sin 2k_n h/2k_n h)]^{1/2}, \tag{4.9}$$

$$\alpha^2 = k^2 - p^2, \qquad \alpha_n^2 = k_n^2 + p^2, \tag{4.10}$$

$\{k_n; n = 1, 2, \ldots\}$ are the real roots of

$$K = -k_n \tan k_n h \tag{4.11}$$

and $k(= ik_0)$ is the real root of

$$K = k\tanh kh. \tag{4.12}$$

Here L is that part of the depth not occupied by the baffle and this approximation may be used for any vertical thin plate; for the surface-piercing baffle L is $a < y < h$. Note that the limit $p \to 0$ does not recover equation (3.1) of Evans and McIver [2] as there is a typographical error in that paper (the factor c_{00}^2 is omitted). Equation (4.6) is readily solved by a standard library foot-finding routine and results for a surface-piercing baffle are given in Figs. 5–6. Figure 5 is for two-dimensional motion and has the same geometry used for Figs 1, 5(a) and 6(a) of [2]; the symmetric $M = 2$ mode is unaffected by the presence of a thin barrier in the plane $x = 0$ and so is represented by a solid straight line in the figure. Similarly, in Fig. 6, the $M = 0$ mode for non-zero p has a fluid motion independent of x and so is also unaffected by the barrier. The variational approximation is reasonable over the whole range of barrier lengths but is particularly good for longer barriers; the variational and small structure approximations are complementary.

Evans & McIver [2] and Watson and Evans [3] also use a wide-spacing approximation to good effect. However, this requires the scattering properties of the internal structure in open water to be known and these are found simply only for a restricted number of geometries.

5. Conclusion

The main results of this paper are simple formulae for the changes in the natural frequencies of fluid in a rectangular tank when some internal structure is introduced. For fully submerged structures these formulae are good approximations over a wide range of parameters. For thin surface-piercing baffles they may be used in conjunction with other approximate solutions to cover the equivalent parameter range. It is noteworthy that the small-structure approximations work well even for the higher modes of oscillation when the length scales of the fluid motion are relatively small.

Other geometries that might be treated by the same method include a ring damper running around the circumference of a cylindrical container and a vertical column standing in a tank of fluid.

Appendix: Submerged multipole potentials

(A) CONSTRUCTION

The aim is to construct solutions of the modified Helmholtz equation that are singular at $(x, y) = (x_0, y_0)$ and satisfy all of the conditions of the problem, equations (2.2.–4), except for the condition on the body contour C. The construction is carried out in three stages. (i) Integral representations are obtained for the fundamental singular solutions of the modified Helmholtz equation, (ii) non-singular terms are added to satisfy the free-surface and bed conditions and (iii) further non-singular terms are added to satisfy the conditions on the vertical walls.

(i) Integral representations of fundamental singularities

Let $(X, Y) = (x - x_0, y - y_0)$ be coordinates measured relative to the singular point. From Twersky [12, equation (31)], for $Y > 0$

$$H_n(kr)i^n e^{in\theta} = \frac{2}{\pi i}\int_0^\infty \frac{e^{-(t^2-k^2)^{1/2}Y}\cos(tX + n\sin^{-1}(t/k))}{(t^2-k^2)^{1/2}}\,dt \tag{A1}$$

where H_n denotes the Hankel function of the first kind and order n. The substitution $k = ip$ gives

$$K_n(pr)e^{in\theta} = \int_0^\infty \frac{e^{-\beta Y}\cos(tX - in\mu)}{\beta}\,dt \tag{A2}$$

where

$$\beta = (p^2 + t^2)^{1/2} \tag{A3}$$

and μ is defined by

$$\sinh\mu = t/p \quad\text{and}\quad \cosh\mu = \beta/p. \tag{A4}$$

Now separate real and imaginary parts, and extend the definition to $Y < 0$ by making use of the relevant symmetry or antisymmetry of each multipole, to obtain

$$K_n(pr)\cos n\theta = (\operatorname{sgn} Y)^n \int_0^\infty \frac{e^{-\beta|Y|}\cos tX\cosh n\mu}{\beta}\,dt, \qquad n = 0, 1, 2\ldots \tag{A5}$$

and

$$K_n(pr)\sin n\theta = (\operatorname{sgn} Y)^{n+1} \int_0^\infty \frac{e^{-\beta|Y|}\sin tX\sinh n\mu}{\beta}\,dt, \qquad n = 1, 2, 3\ldots. \tag{A6}$$

The singularities in equations (A5) and (A6) will be referred to as symmetric and anti-symmetric (about $X = 0$) respectively.

(ii) Free-surface and bed conditicns

To construct symmetric multipoles satisfying the free-surface and bed conditions write

$$\phi_n = K_n(pr)\cos n\theta + \int_0^\infty (A(t)\sinh\beta y + B(t)\cosh\beta y)\frac{\cos tX \cosh n\mu}{\beta}\,\mathrm{d}t. \qquad \text{(A7)}$$

Substitution into the free-surface condition (2.3) and the zero flow condition on $y = h$, and making use of the integral representation (A5) for the singular part, gives simultaneous equations for A and B which when solved yield

$$\begin{aligned}\phi_n &= K_n(pr)\cos n\theta + ⨍_0^\infty \Big\{e^{-\beta(h-y_0)}(K\sinh\beta y - \beta\cosh\beta y) \\ &\quad -(-1)^n(K+\beta)e^{-\beta y_0}\cosh\beta(h-y)\Big\}\frac{\cos tX\cosh n\mu}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\,\mathrm{d}t. \qquad \text{(A8)}\end{aligned}$$

There are poles of the integrand corresponding to the roots of

$$K = \beta\tanh\beta h. \qquad \text{(A9)}$$

Let k be the real positive root of (A9) then the corresponding pole is at $t = (k^2 - p^2)^{1/2}$ which lies on the path of integration for $k > p$. The path of integration is chosen to run beneath this pole in order to give outgoing waves at large distances. If $k < p$ there is no pole on the integration path and the multipoles are non-radiating. These non-radiating multipoles for infinite depth were used by Ursell [13] to construct trapped wave solutions in the presence of a submerged horizontal cylinder.

Alternative forms for ϕ_n follow from replacing $K_n(pr)\cos n\theta$ by the integral representation (A5). For $y > y_0$ the result is

$$\phi_n = ⨍_0^\infty \frac{(K-\beta)e^{\beta y_0} - (-1)^n(K+\beta)e^{-\beta y_0}}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\cosh\beta(h-y)\cos tX\cosh n\mu\,\mathrm{d}t \qquad \text{(A10)}$$

and for $y < y_0$

$$\phi_n = ⨍_0^\infty \frac{e^{-\beta(h-y_0)} + (-1)^n e^{\beta(h-y_0)}}{(K\cosh\beta h - \beta\sinh\beta h)\beta}(K\sinh\beta y - \beta\cosh\beta y)\cos tX\cosh n\mu\,\mathrm{d}t. \qquad \text{(A11)}$$

For $n = 0$ equations (A10–11) are the results for the source solution of the modified Helmholtz equation given by MacCamy [14].

Following a standard procedure (see, for example, Mei [15, p. 380]) the multipole expansions may be expressed as eigenfunction expansions. Thus (A10) is rewritten as

$$\phi_n = \tfrac{1}{2}\int_{-\infty}^\infty \frac{(K-\beta)e^{\beta y_0} - (-1)^n(K+\beta)e^{-\beta y_0}}{(K\cosh\beta h - \beta\sinh\beta h)\beta}\cosh\beta(h-y)e^{it|X|}\cosh n\mu\,\mathrm{d}t \qquad \text{(A12)}$$

where now the path of integration runs below the pole at $t = (k^2 - p^2)^{1/2}$ and above that at $t = -(k^2 - p^2)^{1/2}$ when $k > p$. This integral may be evaluated using the residue theorem.

There are further poles on the imaginary axis in the t-plane corresponding to the imaginary roots of (A9) denoted by $\beta = \pm ik_m$, $m = 1, 2, 3, \ldots$ giving poles at

$$t = \pm i(k_m^2 + p^2)^{1/2} = \pm i\alpha_m. \tag{A13}$$

There are also branch points at $t = \pm ip$ and suitable branch cuts must be inserted that do not cross the $\Re t$-axis, but these do not cause any difficulties. Evaluating the integral with the aid of the closing semi-circular contours described by Mei, modified to circumnavigate the relevant branch cuts, yields

$$\phi_n = \sum_{m=0}^{\infty} \Gamma_{nm} \cos k_m(h-y) e^{-\alpha_m |X|}, \tag{A14}$$

where

$$\Gamma_{nm} = \frac{\pi}{2\alpha_m h N_m^2} (e^{-ik_m(h-y_0)} + (-1)^n e^{ik_m(h-y_0)}) \cosh n\nu_m, \tag{A15}$$

$$N_m^2 = \tfrac{1}{2}\left(1 + \frac{\sin 2k_m h}{2k_m h}\right), \tag{A16}$$

$$k_0 = -ik, \qquad \alpha_0 = -i\alpha = -i(k^2 - p^2)^{1/2} \tag{A17}$$

and ν_m is defined by

$$\sinh \nu_m = \frac{i\alpha_m}{p}, \qquad \cosh \nu_m = \frac{ik_m}{p}. \tag{A18}$$

Equation (A14) is valid throughout the fluid, both (A10) and (A11) yield the same eigenfunction expansion (A14).

Similar calculations may be carried out for multipoles ψ_n that are antisymmetric in X. The form, equivalent to (A8), explicitly displaying the singularity is

$$\begin{aligned} \psi_n = {} & K_n(pr) \sin n\theta + \int_0^{\infty} \Big\{ e^{-\beta(h-y_0)}(K \sinh \beta y - \beta \cosh \beta y) \\ & + (-1)^n (K + \beta) e^{-\beta y_0} \cosh \beta(h-y) \Big\} \frac{\sin tX \sinh n\mu}{(K \cosh \beta h - \beta \sinh \beta h)\beta}\, dt. \end{aligned} \tag{A19}$$

As in (A10–11), the singular part may be incorporated into the integral using (A6). The eigenfunction expansion representation is

$$\psi_n = \operatorname{sgn} X \sum_{m=0}^{\infty} \Delta_{nm} \cos k_m(h-y) e^{-\alpha_m |X|}, \tag{A20}$$

where

$$\Delta_{nm} = \frac{\pi}{2i\alpha_m h N_m^2} (e^{-ik_m(h-y_0)} - (-1)^n e^{ik_m(h-y_0)}) \sinh n\nu_m, \tag{A21}$$

For $k > p$, the first terms in the series (A14) and (A20) give the propagating waves generated by the singularities at large distances.

(iii) Side-wall conditions

To obtain multipole potentials appropriate to a closed basin, that is having zero x-derivative on $x = \pm b$, a similar strategy to that used in (A7) is adopted. Write

$$\phi_n^{(b)} = \phi_n + \sum_{m=0}^{\infty} \Gamma_{nm} \cos k_m(h-y)\{A_m \cosh \alpha_m(x-x_0) + B_m \sinh \alpha_m(x-x_0)\}, \tag{A22}$$

using the eigenfunction representation (A14) for ϕ_n, and apply the boundary conditions on $x = \pm b$ to determine the unknown coefficients. The resulting multipole potentials are

$$\phi_n^{(b)} = \phi_n + \sum_{m=0}^{\infty} \frac{\Gamma_{nm} \cos k_m(h-y)}{\sinh 2\alpha_m b}\{(\cosh 2\alpha_m x_0 + e^{-2\alpha_m b}) \cosh \alpha_m(x-x_0) + \sinh 2\alpha_m x_0 \sinh \alpha_m(x-x_0)\} \tag{A23}$$

and the corresponding result for the antisymmetric multipoles is

$$\psi_n^{(b)} = \psi_n + \sum_{m=0}^{\infty} \frac{\Delta_{nm} \cos k_m(h-y)}{\sinh 2\alpha_m b}\{\sinh 2\alpha_m x_0 \cosh \alpha_m(x-x_0) + (\cosh 2\alpha_m x_0 - e^{-2\alpha_m b}) \sinh \alpha_m(x-x_0)\}. \tag{A24}$$

(B) Expansion about singular point

The generating function for the modified Bessel functions I_q is

$$e^{\frac{1}{2}Z(T+T^{-1})} = \sum_{q=-\infty}^{\infty} T^q I_q(Z). \tag{A25}$$

The substitutions $Z = pr$ and $T = \pm \exp(\mu + i\theta)$, where μ is defined in (A4), give

$$e^{\pm(\beta Y + itX)} = \sum_{q=0}^{\infty} \varepsilon_q (\pm 1)^q \cosh q(\mu + i\theta) I_q(pr) \tag{A26}$$

where $\varepsilon_0 = 1$ and $\varepsilon_q = 2$, $q \geqslant 1$. Equation (A26) may be used to expand the integral terms in (A8) and (A19) to obtain

$$\phi_n - K_n(pr)\cos n\theta = \tfrac{1}{2}\sum_{q=0}^{\infty} \varepsilon_q I_q(pr) \cos q\theta \,⨍_0^{\infty} \Big\{K(e^{2\beta y_0} - (-1)^q) - \beta(e^{2\beta y_0} + (-1)^q) - (-1)^n (K+\beta)(1 + (-1)^q e^{2\beta(h-y_0)})\Big\} \times \frac{e^{-\beta h} \cosh n\mu \cosh q\mu}{(K\cosh\beta h - \beta \sinh \beta h)\beta} \mathrm{d}t \tag{A27}$$

and

$$\psi_n - K_n(pr)\sin n\theta = \sum_{q=1}^{\infty} I_q(pr) \sin q\theta \,⨍_0^{\infty} \Big\{K(e^{2\beta y_0} + (-1)^q)$$

$$-\beta(e^{2\beta y_0} - (-1)^q) + (-1)^n (K + \beta)(1 - (-1)^q e^{2\beta(h-y_0)})\Big\}$$
$$\times \frac{e^{-\beta h} \sinh n\mu \sinh q\mu}{(K \cosh \beta h - \beta \sinh \beta h)\beta} dt. \tag{A28}$$

These expansions are valid for $0 < r < 2y_0$. As $h \to \infty$ in (A27) the result of Ursell [13, equation (11)] is recovered.

To expand the summation terms in (A23) and (A24) a modification of the result (A26) is needed. Replace t by $i(k_m^2 + p^2)^{1/2} = i\alpha_m$ to give

$$e^{\pm(ik_m Y - \alpha_m X)} = \sum_{q=0}^{\infty} \varepsilon_q (\pm 1)^q \cosh q(\nu_m + i\theta) I_q(pr) \tag{A29}$$

where ν_m is defined by

$$\sinh \nu_m = i\alpha_m / p \text{ and } \cosh \nu_m = ik_m / p. \tag{A30}$$

Thus

$$\phi_n^{(b)} - \phi_n = \tfrac{1}{2} \sum_{q=0}^{\infty} \varepsilon_q I_q(pr) \Big\{ \cos q\theta \sum_{m=0}^{\infty} \frac{\Gamma_{nm}(\cosh 2\alpha_m x_0 + e^{-2\alpha_m b})}{\sinh 2\alpha_m b}$$
$$\times ((-1)^q e^{ik_m(h-y_0)} + e^{-ik_m(h-y_0)}) \cosh q\nu_m + i \sin q\theta \sum_{m=0}^{\infty}$$
$$\times \frac{\Gamma_{nm} \sinh 2\alpha_m x_0}{\sinh 2\alpha_m b} ((-1)^q e^{ik_m(h-y_0)} - e^{-ik_m(h-y_0)}) \sinh q\nu_m \Big\} \tag{A31}$$

and

$$\psi_n^{(b)} - \psi_n = \tfrac{1}{2} \sum_{q=0}^{\infty} \varepsilon_q I_q(pr) \Big\{ \cos q\theta \sum_{m=0}^{\infty} \frac{\Delta_{nm} \sinh 2\alpha_m x_0}{\sinh 2\alpha_m b} ((-1)^q e^{ik_m(h-y_0)}$$
$$+ e^{-ik_m(h-y_0)}) \cosh q\nu_m + i \sin q\theta \sum_{m=0}^{\infty} \frac{\Delta_{nm}(\cosh 2\alpha_m x_0 - e^{-2\alpha_m b})}{\sinh 2\alpha_m b}$$
$$\times ((-1)^q e^{ik_m(h-y_0)} - e^{-ik_m(h-y_0)}) \sinh q\nu_m \Big\}. \tag{A32}$$

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